

CHAIN POLYTOPES AND ALGEBRAS WITH STRAIGHTENING LAWS

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ABSTRACT. It will be shown that the toric ring of the chain polytope of a finite partially ordered set is an algebra with straightening laws on a finite distributive lattice. Thus in particular every chain polytope possesses a regular unimodular triangulation arising from a flag complex. Moreover, the problem when the order polytope and the chain polytope of a finite partially ordered set are combinatorially (or unimodularly) equivalent will be studied.

INTRODUCTION

In [6], the order polytope $\mathcal{O}(P)$ and the chain polytope $\mathcal{C}(P)$ of a finite poset (partially ordered set) P are studied in detail from a view point of combinatorics. Toric rings of order polytopes are studied in [1]. In particular it is shown that the toric ring $K[\mathcal{O}(P)]$ of the order polytope $\mathcal{O}(P)$ is an algebra with straightening laws ([2, p. 124]) on a finite distributive lattice. In the present paper, it will be proved that the toric ring $K[\mathcal{C}(P)]$ of the chain polytope $\mathcal{C}(P)$ is also an algebra with straightening laws on a finite distributive lattice. It then follows immediately that $\mathcal{C}(P)$ possesses a regular unimodular triangulation arising from a flag complex. Moreover, the problem when $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are combinatorially (or unimodularly) equivalent will be studied.

1. TORIC RINGS OF ORDER POLYTOPES AND CHAIN POLYTOPES

Let $P = \{x_1, \dots, x_d\}$ be a finite poset. For each subset $W \subset P$, we associate $\rho(W) = \sum_{i \in W} \mathbf{e}_i \in \mathbb{R}^d$, where $\mathbf{e}_1, \dots, \mathbf{e}_d$ are the unit coordinate vectors of \mathbb{R}^d . In particular $\rho(\emptyset)$ is the origin of \mathbb{R}^d . A *poset ideal* of P is a subset I of P such that, for all x_i and x_j with $x_i \in I$ and $x_j \leq x_i$, one has $x_j \in I$. An *antichain* of P is a subset A of P such that x_i and x_j belonging to A with $i \neq j$ are incomparable.

Recall that the *order polytope* is the convex polytope $\mathcal{O}(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \dots, a_d) \in \mathbb{R}^d$ such that $0 \leq a_i \leq 1$ for every $1 \leq i \leq d$ together with $a_i \geq a_j$ if $x_i \leq x_j$ in P . The vertices of $\mathcal{O}(P)$ is those $\rho(I)$ such that I is a poset ideal of P ([6, Corollary 1.3]). The *chain polytope* is the convex polytope $\mathcal{C}(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \dots, a_d) \in \mathbb{R}^d$ such that $a_i \geq 0$ for every $1 \leq i \leq d$ together with

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} \leq 1$$

for every maximal chain $x_{i_1} < x_{i_2} < \dots < x_{i_k}$ of P . The vertices of $\mathcal{C}(P)$ is those $\rho(A)$ such that A is an antichain of P ([6, Theorem 2.2]).

Let $S = K[x_1, \dots, x_d, t]$ denote the polynomial ring over a field K whose variables are the elements of P together with the new variable t . For each subset $W \subset P$, we associate

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the squarefree monomial $x(W) = \prod_{i \in W} x_i \in S$. In particular $x(\emptyset) = 1$. The *toric ring* $K[\mathcal{O}(P)]$ of $\mathcal{O}(P)$ is the subring of R generated by those monomials $t \cdot x(I)$ such that I is a poset ideal of P . The toric ring $K[\mathcal{C}(P)]$ of $\mathcal{C}(P)$ is the subring of R generated by those monomials $t \cdot x(A)$ such that A is an antichain of P .

2. ALGEBRAS WITH STRAIGHTENING LAWS

Let $R = \bigoplus_{n \geq 0} R_n$ be a graded algebra over a field $R_0 = K$. Suppose that P is a poset with an injection $\varphi : P \rightarrow R$ such that $\varphi(\alpha)$ is a homogeneous element of R with $\deg \varphi(\alpha) \geq 1$ for every $\alpha \in P$. A *standard monomial* of R is a finite product of the form $\varphi(\alpha_1)\varphi(\alpha_2)\cdots$ with $\alpha_1 \leq \alpha_2 \leq \cdots$. Then we say that $R = \bigoplus_{n \geq 0} R_n$ is an *algebra with straightening laws* on P over K if the following conditions are satisfied:

- The set of standard monomials is a basis of R as a vector space over K ;
- If α and β in P are incomparable and if

$$(1) \quad \varphi(\alpha)\varphi(\beta) = \sum_i r_i \varphi(\gamma_i) \varphi(\gamma_2) \cdots,$$

where $0 \neq r_i \in K$ and $\gamma_1 \leq \gamma_2 \leq \cdots$, is the unique expression for $\varphi(\alpha)\varphi(\beta) \in R$ as a linear combination of distinct standard monomials, then $\gamma_1 \leq \alpha$ and $\gamma_1 \leq \beta$ for every i .

We refer the reader to [2, Chapter XIII] for fundamental materials on algebras with straightening laws. The relations (1) are called the *straightening relations* of R .

Let P be an arbitrary finite poset and $\mathcal{J}(P)$ the finite distributive lattice ([7, p. 252]) consisting of all poset ideals of P , ordered by inclusion. The toric ring $K[\mathcal{O}(P)]$ of the order polytope $\mathcal{O}(P)$ can be a graded ring with $\deg(t \cdot x(I)) = 1$ for every $I \in \mathcal{J}(P)$. We then define the injection $\varphi : \mathcal{J}(P) \rightarrow K[\mathcal{O}(P)]$ by setting $\varphi(I) = t \cdot x(I)$ for every $I \in \mathcal{J}(P)$. One of the fundamental results obtained in [1] is that $K[\mathcal{O}(P)]$ is an algebra with straightening laws on $\mathcal{J}(P)$. Its straightening relations are

$$(2) \quad \varphi(I)\varphi(J) = \varphi(I \cap J)\varphi(I \cup J),$$

where I and J are poset ideals of P which are incomparable in $\mathcal{J}(P)$.

Theorem 2.1. *The toric ring of the chain polytope of a finite poset is an algebra with straightening laws on a finite distributive lattice.*

Proof. Let P be an arbitrary finite poset and $\mathcal{C}(P)$ its chain polytope. The toric ring $K[\mathcal{C}(P)]$ can be a graded ring with $\deg(t \cdot x(A)) = 1$ for every antichain A of P . For a subset $Z \subset P$, we write $\max(Z)$ for the set of maximal elements of Z . In particular $\max(Z)$ is an antichain of P . The poset ideal of P generated by a subset $Y \subset P$ is the smallest poset ideal of P which contains Y .

Now, we define the injection $\psi : \mathcal{J}(P) \rightarrow K[\mathcal{C}(P)]$ by setting $\psi(I) = t \cdot x(\max(I))$ for all poset ideal I of P . If I and J are poset ideals of P , then

$$(3) \quad \psi(I)\psi(J) = \psi(I \cup J)\psi(I * J),$$

where $I * J$ is the poset ideal of P generated by $\max(I \cap J) \cap (\max(I) \cup \max(J))$. Since $I * J \subset I$ and $I * J \subset J$, the relations (3) satisfy the condition of the straightening relations.

It remains to prove that the set of standard monomials of $K[\mathcal{C}(P)]$ is a K -basis of $K[\mathcal{C}(P)]$. It follows from [6, Theorem 4.1] that the Hilbert function ([2, p. 33]) of the Ehrhart ring ([2, p. 97]) of $\mathcal{O}(P)$ coincides with that of $\mathcal{C}(P)$. Since $\mathcal{O}(P)$ and $\mathcal{C}(P)$ possess the integer decomposition property ([4, Lemma 2.1]), the Ehrhart ring of $\mathcal{O}(P)$ coincides with $K[\mathcal{O}(P)]$ and the Ehrhart ring of $\mathcal{C}(P)$ coincides with $K[\mathcal{C}(P)]$. Hence the Hilbert function of $K[\mathcal{O}(P)]$ is equal to that of $K[\mathcal{C}(P)]$. Thus the set of standard monomials of $K[\mathcal{C}(P)]$ is the K -basis of $K[\mathcal{C}(P)]$, as desired. \square

3. FLAG AND UNIMODULAR TRIANGULATIONS

The fact that $K[\mathcal{C}(P)]$ is an algebra with straightening laws guarantees that the toric ideal of $\mathcal{C}(P)$ possesses an initial ideal generated by squarefree quadratic monomials. We refer the reader to [3] and [5, Appendix] for the background of the existence of squarefree quadratic initial ideals of toric ideals. By virtue of [8, Theorem 8.3], it follows that

Corollary 3.1. *Every chain polytope possesses a regular unimodular triangulation arising from a flag complex.*

4. AFFINELY EQUIVALENCE OF ORDER POLYTOPES AND CHAIN POLYTOPES

Since both $K[\mathcal{O}(P)]$ and $K[\mathcal{C}(P)]$ are algebras with straightening laws on the distributive lattice $\mathcal{J}(P)$, it is natural to ask when the straightening relations of $K[\mathcal{O}(P)]$ and $K[\mathcal{C}(P)]$ coincide. It follows from (2) and (3) that the straightening relations of $K[\mathcal{O}(P)]$ and $K[\mathcal{C}(P)]$ coincide if and only if $\psi(I * J) = \psi(I \cap J)$ for all I and J belonging to $\mathcal{J}(P)$. Moreover, one has $\psi(I * J) = \psi(I \cap J)$ if and only if $\max(I \cap J) \subset \max(I) \cup \max(J)$.

Let P be a disjoint union of rooted trees ([7, p. 572]). Then every poset ideal of P is also a disjoint union of rooted trees. If $I \subset P$ and $J \subset P$ are rooted trees with $I \cap J \neq \emptyset$, then one has either $I \subset J$ or $J \subset I$. Hence, for all poset ideals I and J of P , one has

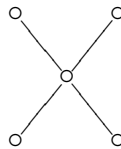
$$\max(I \cap J) = (\max(I) \cup \max(J)) \setminus \max(I \cup J).$$

Thus in particular $\max(I \cap J) \subset \max(I) \cup \max(J)$.

Lemma 4.1. *If the straightening relations of $K[\mathcal{O}(P)]$ and $K[\mathcal{C}(P)]$ coincide, then $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are affinely equivalent and hence combinatorially equivalent.*

Corollary 4.2. *Suppose that P is a disjoint union of rooted trees. Then $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are affinely equivalent and hence combinatorially equivalent.*

There exists a poset whose order polytope and chain polytope are not combinatorially equivalent. For example, the following poset P is from [6, p. 13], where $\mathcal{O}(P)$ has eight facets and $\mathcal{C}(P)$ has nine facets.



5. UNIMODULAR EQUIVALENCE

Let $\mathbb{Z}^{d \times d}$ denote the set of $d \times d$ integral matrices. Recall that a matrix $A \in \mathbb{Z}^{d \times d}$ is *unimodular* if $\det(A) = \pm 1$. Given integral polytopes $\mathcal{P} \subset \mathbb{R}^d$ of dimension d and $\mathcal{Q} \subset \mathbb{R}^d$ of dimension d , we say that \mathcal{P} and \mathcal{Q} are *unimodularly equivalent* if there exist a unimodular matrix $U \in \mathbb{Z}^{d \times d}$ and an integral vector $\mathbf{w} \in \mathbb{Z}^d$ such that $\mathcal{Q} = f_U(\mathcal{P}) + \mathbf{w}$, where f_U is the linear transformation of \mathbb{R}^d defined by U , i.e., $f_U(\mathbf{v}) = \mathbf{v}U$ for all $\mathbf{v} \in \mathbb{R}^d$.

Now, we wish to discuss the problem when $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are unimodularly equivalent. Theorems 5.1 and 5.2 strengthen [6, Theorem 2.3] and Corollary 4.2, respectively.

Theorem 5.1. *Suppose that P has length at most one, i.e., P has no three-element chain. Then $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are unimodularly equivalent.*

Proof. If the rank of P is 0, i.e., P is an antichain, then every subset of P is both a poset ideal and an antichain. In particular $\mathcal{O}(P)$ and $\mathcal{C}(P)$ coincides.

Suppose that $P = \{x_1, \dots, x_k, y_1, \dots, y_s\}$ is of length one, where the x_i 's are minimal elements of P . Let $p = (a_1, \dots, a_k, b_1, \dots, b_s) \in \mathbb{R}^{k+s}$. Define the map f by sending p to $f(p) = (1 - a_1, \dots, 1 - a_k, b_1, \dots, b_s)$. Let U be the diagonal matrix with k -1 's and s 1 's, and $w = (1, \dots, 1, 0, \dots, 0)$ with k 1 's and s 0 's. Clearly U is unimodular and $f(p) = pU + w$. Since $a_i \geq b_j$ if and only if $(1 - a_i) + b_j \leq 1$, it is a routine work to show that f is a bijection between $\mathcal{O}(P)$ and $\mathcal{C}(P)$. \square

Theorem 5.2. *Suppose that P is a disjoint union of rooted trees. Then $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are unimodularly equivalent.*

Proof. Let $P = \{x_1, \dots, x_n\}$, where $i < j$ if $x_i < x_j$, be a disjoint union of rooted trees and define the map $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as follows: One has $g(p_1, \dots, p_n) = (q_1, \dots, q_n)$, where

- if x_i is a root, then $q_i = p_i$;
- if x_i is not a root and x_j is the parent of x_i , then $q_i = p_i - p_j$.

It follows that g is a bijection between $\mathcal{O}(P)$ and $\mathcal{C}(P)$. If $g(p) = pU$ with $U \in \mathbb{Z}^{d \times d}$, then U is lower triangular with 1 's and -1 's on its diagonal. Thus g is unimodular. \square

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